LS-9
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The Effects of Trapped Ions in an Electron Storage Ring

The fast electrons of the beam will ionize the residual gas molecules. The detached electron will be lost on the vacuum chamber wall. The slow ions will be focused when they are in the potential well of the electron beam and defocused during the remaining time. The equations of motion of the ions may be written in the form

$$\frac{d^2x}{dt^2} = -\omega_x^2(t)\ x$$

$$\frac{d^2z}{dt^2} = -\omega_z^2(t)\ z.$$

where $\omega_x^2 = \frac{(n_e-n_i)e^2}{\varepsilon_o\ M} \frac{b}{a+b}$, $\omega_z^2 = -\frac{(n_e-n_i)e^2}{\varepsilon_o M} \frac{a}{a+b}$

when the ions are in the potential well of the electron beam and

$$\omega_x^2 = -\frac{n_i e^2}{\varepsilon_o\ M} \frac{b}{a+b}\ ,\quad \omega_z^2 = -\frac{n_i e^2}{\varepsilon_o\ M} \frac{a}{a+b}$$

when the ions are between the electron bunches.

Notation:

n_e = electron density

n_i = ion density

a = radial beam size

b = vertical beam size

M = ion mass

$\varepsilon_o = \frac{1}{36\pi}10^{-9}$

In general, the vertical emittance is smaller than the radial emittance. For a real value of the transition energy, the radial damping time is longer than the vertical damping time. (Damping partition numbers: $J_x = 1 - \frac{1}{\gamma_t^2}\frac{R}{\rho}$, $J_z = 1$.)

Therefore, in what follows, one will consider only the vertical motion and delete the subscript z. The quantity $\omega(t)$ is a periodic function of time with the bunch periodicity. The ions remain inside the beam if their motion in the alternating space-charge fields is stable. The phase advance σ per bunch period is given by the relation

$$\cos\sigma = \cos\omega_1\tau_1 \cosh\omega_2(\tau-\tau_1) - \frac{\omega_1^2-\omega_2^2}{2\omega_1\omega_2} \sin\omega_1\tau_1 \sinh\omega_2(\tau-\tau_1),$$

where $\omega_1^2 = \frac{(n_e - n_i)e^2}{\varepsilon_o M}\frac{a}{a+b}$ and $\omega_2^2 = \frac{n_i e^2}{\varepsilon_o M}\frac{a}{a+b}$

τ_1 = bunch length and τ = bunch period.

Two cases will be considered.

1. $\tau-\tau_1 \ll \tau$ (coasting beam approximation).

Taking into account that in the beginning the ion density can be neglected, one obtains the trapping condition

$$\cos\sigma > -1 \text{ or } \frac{n_e\, e^2 \tau_1 \tau}{\varepsilon_o M} \frac{a}{a+b} < \pi^2$$

Substitutions of $r_e = \frac{e^2}{4\pi\varepsilon_o m_e c^2}$ and $n_e = \frac{N_e \tau/\tau_1}{2\pi R\ \pi\ ab}$, where m_e is the mass of an electron and N_e is the number of electrons in the ring, give the trapping condition

$$\frac{M}{m_e} > \frac{8\ N_e\ Rr_e}{m_b^2\ \pi b(a+b)} = \left(\frac{4\ R}{m_b}\right)^2 \frac{I}{b(a+b)\ I_A}, \qquad (1)$$

where m_b (= 1, 2, 3...h) is the number of bunches, I is the average current in the ring and $I_A = \frac{ec}{r_e}$ (≈ 17000 A) and h is the harmonic number.

2. $\tau_1 << \tau$ (thin lens approximation).

In this case the condition $\cos\sigma > -1$ gives $1 - 1/2\ \frac{n_e \tau_1 \tau\ e^2}{\varepsilon_o M} \frac{a}{a+b} > -1$ or

$$\frac{M}{m_e} > \frac{2\pi R r_e N_e}{b(a+b) m_b^2} = \left(\frac{2\pi R}{m_b}\right)^2 \frac{I}{b(a+b) I_A} \qquad (2)$$

In deriving Eqs. (1) and (2), it is assumed that the bunches are equally spaced. As the ion density increases, σ decreases. The maximum ion density is determined by the condition cos σ = 1 which is approximately equivalent to an ion density equal to the average electron density $n_i = \bar{n}_e$. This state can be reached only when the damping time is much longer than the neutralization time defined by

$$\tau_n = \Delta E_i / c\ \frac{dE}{dx}, \qquad (3)$$

where ΔE_i = the average ionization energy

c = velocity of light

$\frac{dE}{dx}$ = the energy loss per unit length, approximately given by

$$\frac{dE}{dx} = 2\pi r_e^2 \, n_M \, Z \, m_e c^2 \, \ln \frac{2.87 \times 10^9 \gamma^3}{Z^2} \ \mathrm{eV/m} \tag{4}$$

where $n_M = 2.687 \times 10^{25} \frac{p}{760}$ and Z = number of electrons per molecule.

Example:

During the injection in the Aladdin storage ring, one may have a fully neutralized electron beam (number of electrons in the ring N_e = number of ions in the ring N_i).

Injection energy 100 MeV (γ = 196.7)
Circumference $2\pi R$ = 88 m (R = 14 m)
Magnet radius ρ = 2.08 m

Assuming the residual gas is CO, one finds $\frac{M}{m_e} = 5.1 \times 10^4$ and the trapping condition is satisfied for $I < 100$ mA, $b(a + b) > 10^{-6}$ m^2 and $m_b > 1$. Using the damping partition number $J_z = 1$ and $J_x = 1 - \frac{R}{\gamma_t^2 \rho}$ with $\gamma_t = 5.3$ one obtains τ_x = 18 sec and τ_z = 14 sec

Eqs. (3) and (4) give for ΔE_i = 50 eV

$p = 10^{-9}$ mm Hg	τ_n = 0.42 sec
$p = 10^{-8}$ mm Hg	τ_n = 0.042 sec

The damping time is much longer than the neutralization time and one may assume that

$$N_i \approx N_e = \frac{I \, T_o}{e} \quad (T_o = \text{rotation time})$$

The positive betatron tune shift is given by

$$\nu_z^2 = \nu_{zo}^2 + \frac{1}{\gamma}\frac{2R\ r_e N_i}{\pi b(a+b)}$$

Substitution of $\nu_z = 7.5$ and $\nu_{zo} = 7.15$ give

$$\frac{1}{\gamma}\frac{2R\ r_e\ N_i}{\pi b(a+b)} = \frac{1}{\gamma}\frac{(2R)^2\ I}{b(a+b)\ I_A} \approx 5 \text{ or } \frac{I}{b(a+b)} = 2 \times 10^4$$

The time dependence of the beam sizes a and b can be written in for form

$$a = \Delta a + a_o\ e^{-\frac{t}{18}}$$

$$b = \Delta b + b_o\ e^{-\frac{t}{14}}$$

A fraction of the beam will be lost if

$$\sum_{k=0}^{k_1} \frac{\Delta I_k}{(a_k+b_k)b_k} > 2 \times 10^4,$$

Assuming injection every 5 seconds and a net injected current of 1 mA, one has

$$\Delta\ I_k = 0.001\ \text{A}$$

$$a_k = \Delta a + a_o\ e^{-\frac{t-5k}{18}}$$

$$b_k = \Delta b + b_o\ e^{-\frac{t-5k}{14}}$$

Figure 1 gives $\sum_{k=0}^{8} \Delta I_k$ during the injection under the following assumptions:

$a_o = 5$ mm, $\Delta a = 0.3$ mm, $b_o = 1.6$ mm and $\Delta b = 0.1$ mm.

Dotted line : uniform distribution

Solid line : parabolic distribution

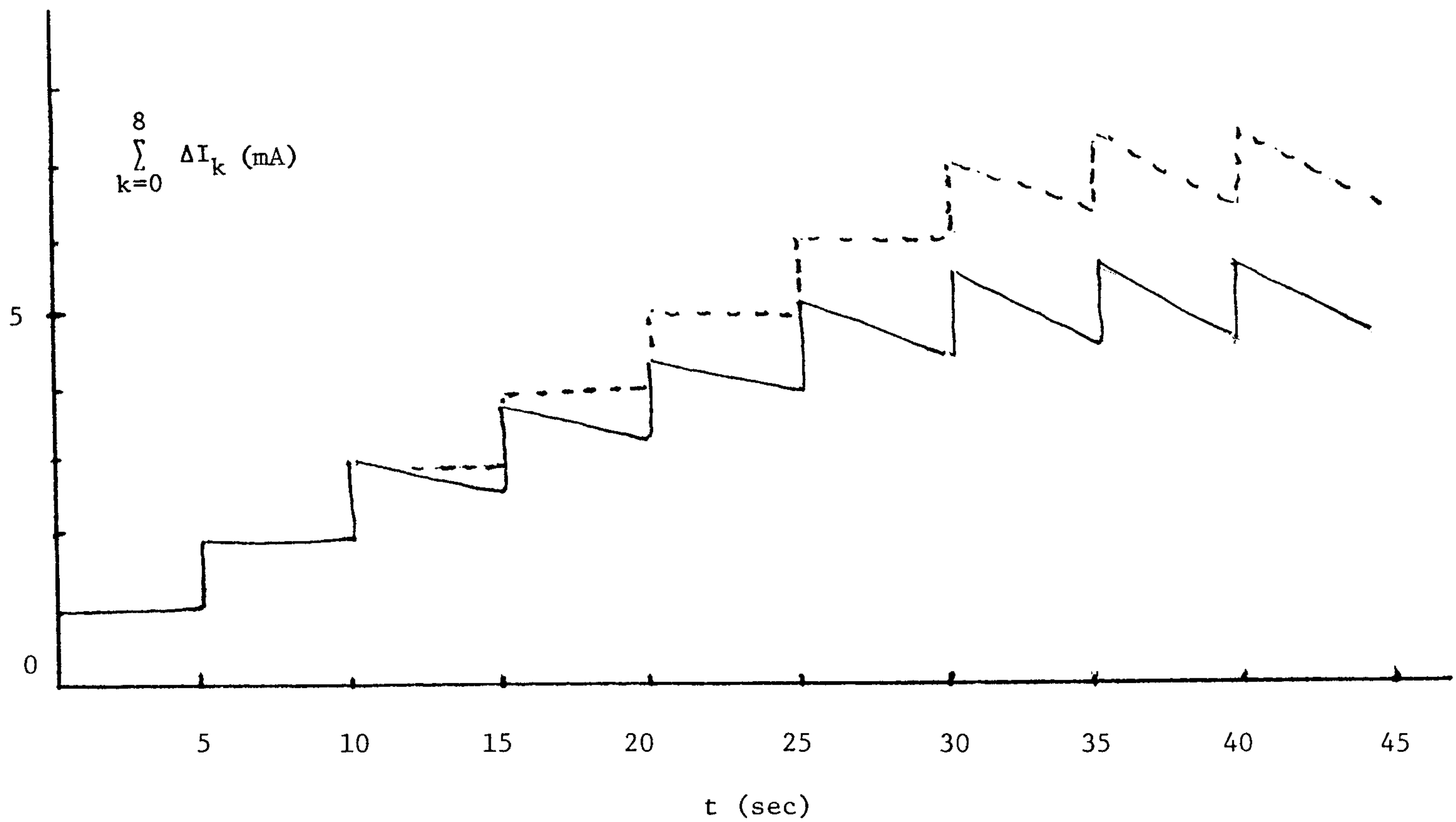

Figure 1. The injected current as a function of time.

Under these conditions, the maximum current is about 7 mA for the uniform distribution and 5 mA for the parabolic distribution.

In most electron storage rings, the damping time is shorter than the neutralization time. In what follows, an attempt will be made to explain why in these storage rings

1. The number of ions in the ring $N_i \ll N_e$
2. Actual beam size is larger than predicted by theory.

The trapping condition is satisfied, except possibly for operation with one bunch in the ring. The beam envelop equations can be written in the form

$$\frac{d^2 b_e}{dt^2} + \left(\nu_{zo}^{\ 2} + \delta\nu_e^2\right) \omega_o^2 \, b_e - \frac{\varepsilon_e^2}{b_e^3} = 0,$$

$$\frac{d^2 b_i}{dt^2} + \nu_i^2 \, \omega_o^2 \, b_i - \frac{\varepsilon_i^2}{b_i^3} = 0,$$

where $\delta\nu_e{}^2 = \dfrac{2R \; r_e \; N_i}{\gamma\pi b_i (a_i + b_i)}$

$$\nu_i{}^2 = \frac{m_e}{M} \frac{2Rr_eN_e}{\pi b_e(a_e + b_e)} = \frac{m_e}{M} \frac{(2R)^2 I}{b_e(a_e + b_e) \; I_A}$$

ε_e = electron emittance

ε_i = ion emittance

For the unperturbed motion, one has $b_e = b_{eo}$, $b_i = b_{io}$, $a_e = a_i = a$ with b_{eo} and b_{io} satisfying the following relations

$$\left(\nu_{zo}^{\ 2} + \delta \; \nu_{eo}{}^2\right) \omega_o^2 \; b_{eo} - \frac{\varepsilon_e^{\ 2}}{b_{eo}^{\ 3}} = 0$$

$$\nu_i^2 \; \omega_o^2 \; b_{io} - \frac{\varepsilon_i^{\ 2}}{b_{io}^{\ 3}} = 0 \qquad (5)$$

Substituting

$$b_e = b_{eo} + \eta_e \text{ and } b_i = b_{io} + \eta_i,$$

retaining the linear terms only and using Eq. (5) one obtains

$$\frac{d^2\eta_e}{dt^2} + \left\{4(\nu_{zo}^{\ 2} + \delta\nu_e^2) \; \eta_e - \delta\nu_i^2 \; \eta_i\right\}\omega_o^2 = 0,$$

$$\frac{d^2\eta_i}{dt^2} + (4\nu_i^2\,\eta_i - \nu_e^2\eta_e)\omega_o^2 = 0 \ ,$$

where
$$\delta\nu_i^2 = \frac{b_{eo}}{b_{io}}\,\delta\nu_e^2 \approx \delta\nu_e^2$$

$$\nu_e^2 = \frac{b_{io}}{b_{eo}}\,\nu_i^2 \approx \nu_i^2$$

It can be verified by substitution that, if η_e and η_i can be written in the form

$$\eta_e = \eta_{eo}e^{j(\omega t - n\theta)} \ , \qquad \frac{d\theta}{dt} = \omega_o$$

$$\eta_i = \eta_{io}e^{j(\omega t - n\theta)} \ , \qquad \frac{d\theta}{dt} = 0$$

then the dispersion equation is given by

$$\{-(\omega - n\omega_o)^2 + 4\,(\nu_{zo}^{\ 2} + \delta\nu_e^2)\}\{-\omega^2 + 4\nu_i^2\} - \omega_o^4\ \delta\nu_e^2\nu_i^2 = 0$$

In this equation, $\delta\nu_i^2$ and ν_e^2 are replaced by $\delta\nu_e^2$ and ν_i^2. Introducing $\mu = \frac{\omega}{\omega_o}$, one can rewrite the dispersion relation in the form

$$\frac{\delta\nu_e^2\ \nu_i^2}{\{(\mu - n)^2 - 4\nu_z^2\}\{\mu^2 - 4\nu_i^2\}} = 1 \ , \tag{6}$$

$$\text{where } \nu_z^2 = \nu_{zo}^{\ 2} + \delta\nu_e^2 \ .$$

Rewriting Eq. (6) in the form $f(\mu) = 1$, where

$$f(\mu) = \frac{1/4\ \delta\ \nu_e^2}{(\mu-n)^2 - (2\nu_z^1)^2} + \frac{4\nu_i^2}{\mu^2}\ ,\quad (\nu_z^1)^2 = \nu_{zo}^{\ 2} + \frac{15}{16}\,\delta\nu_e^2$$

One sees that for $n < 2\nu_z^1$, there are 4 real roots of $f(\mu) = 1$ and, therefore, the motion is stable.

For $n > 2\nu_z$, the motion is unstable if

$$|n - 2\nu_z - 2\nu_i| < \frac{\delta\nu_e}{2}\sqrt{\frac{\nu_i}{\nu_z}} \tag{7}$$

The growth rate of this instability is given by

$$\frac{1}{2}\,\omega_o\left[\frac{\delta\nu_e^2\nu_i}{4\nu_z} - (n - 2\nu_z - 2\nu_i)^2\right]^{1/2} \tag{8}$$

The electron beam size decreases after injection and ν_i increases. As soon as Eq. (7) is satisfied, both beams become unstable. The beams increase in size and the resonance condition (Eq. 7) is not satisfied anymore. One has damping again and the process repeats. The result is a beam size pulsating with time.

If the damping time is not much smaller than the neutralization time, a steady state can be reached such that the growth rate (Eq. 8) is approximately equal to the radiation damping rate. In this case the beam size, the current and the ion density will be determined by the relation

$$\omega_o\left[\frac{\delta\nu_e^2\ \nu_i}{4\ \nu_z} - \left(n - 2\ \nu_z - 2\ \nu_i\right)^2\right]^{1/2} = \frac{2}{\tau_z}\ .$$